

THE C^* -ALGEBRA GENERATED BY IRREDUCIBLE TOEPLITZ AND COMPOSITION OPERATORS

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ABSTRACT. We describe the C^* -algebra generated by an irreducible Toeplitz operator T_ψ , with continuous symbol ψ on the unit circle \mathbb{T} , and finitely many composition operators on the Hardy space H^2 induced by certain linear-fractional self-maps of the unit disc, modulo the ideal of compact operators $K(H^2)$. For composition operators with automorphism symbols, we show that this algebra is not isomorphic to the one generated by the shift and composition operators.

1. INTRODUCTION

The Hardy space $H^2 = H^2(\mathbb{D})$ is the collection of all analytic functions f on the open unit disk \mathbb{D} satisfying the norm condition

$$\|f\|^2 := \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

For any analytic self-map φ of the open unit disk \mathbb{D} , a bounded composition operator on H^2 is defined by

$$C_\varphi : H^2 \rightarrow H^2, \quad C_\varphi(f) = f \circ \varphi.$$

If $f \in H^2$, then the radial limit $f(e^{i\theta}) := \lim_{r \rightarrow 1} f(re^{i\theta})$ exists almost everywhere on the unit circle \mathbb{T} . Hence we can consider H^2 as a subspace of $L^2(\mathbb{T})$. Let ϕ be a bounded measurable function on \mathbb{T} and P_{H^2} be the orthogonal projection of $L^2(\mathbb{T})$ (associated with normalized arc-length measure on \mathbb{T}) onto H^2 . The Toeplitz operator T_ϕ is defined on H^2 by $T_\phi f = P_{H^2}(\phi f)$ for all $f \in H^2$. Coburn in [4, 5] shows that the unital C^* -algebra $C^*(T_z)$ generated by the unilateral shift operator T_z contains compact operators on H^2 as an ideal and every element $a \in C^*(T_z)$ has a unique representation $a = T_\phi + k$ for some $\phi \in C(\mathbb{T})$ and $k \in \mathfrak{K} := K(H^2)$. He shows that $C^*(T_z)/\mathfrak{K}$ is $*$ -isomorphic to $C(\mathbb{T})$, and determines essential spectrum of Toeplitz operators with continuous symbol.

Recently the unital C^* -algebra generated by the shift operator T_z and the composition operator C_φ for a linear-fractional self-map φ of \mathbb{D} is studied. For a linear-fractional self-map φ on \mathbb{D} , if $\|\varphi\|_\infty < 1$ then C_φ is a compact operator on H^2 [15]. Therefore one should consider those linear-fractional self-maps φ which satisfy $\|\varphi\|_\infty = 1$. If moreover φ is an automorphism of \mathbb{D} , then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the crossed product $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ [7, 8]. When φ is not an automorphism there are three deferent cases:

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- (i) φ has only one fixed point γ which is on the unit circle \mathbb{T} (i.e. φ is a parabolic map). In this case $C^*(T_z, C_\varphi)/\mathfrak{K}$ is a commutative C^* -algebra isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus C_0([0, 1])$, where $C_\gamma(\mathbb{T})$ is the set of functions in $C(\mathbb{T})$ vanishing at $\gamma \in \mathbb{T}$ and $C_0([0, 1])$ is the set of all $f \in C([0, 1])$ vanishing at zero [14].
- (ii) φ has a fixed point $\gamma \in \mathbb{T}$ and fixes another point in $\mathbb{C} \cup \{\infty\}$ (equivalently φ has a fixed point $\gamma \in \mathbb{T}$ and $\varphi'(\gamma) \neq 1$). In this case $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus (C_0([0, 1]) \rtimes \mathbb{Z})$ [14].
- (iii) φ fixes no point of \mathbb{T} but there exist distinct points $\gamma, \eta \in \mathbb{T}$ with $\varphi(\gamma) = \eta$. In this case $C^*(T_z, C_\varphi)/\mathfrak{K}$ is the C^* -subalgebra \mathcal{D} of $C(\mathbb{T}) \oplus M_2(C([0, 1]))$ defined by

$$\mathcal{D} = \left\{ (f, V) \in C(\mathbb{T}) \oplus M_2(C([0, 1])) : V(0) = \begin{bmatrix} f(\gamma) & 0 \\ 0 & f(\eta) \end{bmatrix} \right\}$$

[9].

This paper generalizes the above results. The generalization is two fold. We replace the shift operator T_z by an irreducible Toeplitz operator T_ψ with continuous symbol ψ on \mathbb{T} , and a single composition operator with finitely many composition operators on the Hardy space H^2 induced by certain linear-fractional self-maps of \mathbb{D} . The paper is organized as follows. In section 2 we review basic facts and known results which are used later in the paper. In section 3 we find the C^* -algebra $C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$, where $\varphi_1, \dots, \varphi_n$ are as in the case (ii). In section 4 we replace the shift operator T_z by an irreducible Toeplitz operator T_ψ with symbol $\psi \in C(\mathbb{T})$ and obtain more general results in the above cases. When φ is an automorphism, the composition operator C_φ often generates the unilateral shift operator T_z . We investigate this case in section 5.

2. PRELIMINARIES

In this section we review some of the known results which are used in the next sections. Here we use $C_0([0, 1])$ to denote the set of functions in $C([0, 1])$ vanishing at zero and $[T]$ to denote the coset of operator $T \in B(H^2)$ in the Calkin algebra $B(H^2)/K(H^2)$.

A linear-fractional self-map ρ with fixed point $\gamma \in \mathbb{T}$ is *parabolic* if and only if $\rho'(\gamma) = 1$. In this case, ρ is conjugate to a translation on the right half plan $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ via the conformal map $\alpha : z \mapsto (\gamma + z)/(\gamma - z)$ of \mathbb{D} onto Ω . Therefore $\alpha \circ \rho \circ \alpha^{-1}$ is the translation map $z \mapsto z + a$ for some $a \in \mathbb{C}$ with non-negative real part. We denote the map ρ by $\rho_{\gamma, a}$. This is an automorphism of \mathbb{D} if and only if $\operatorname{Re} a = 0$. For $\gamma \in \mathbb{T}$, the set $\mathbb{P}_\gamma := \{C_{\rho_{\gamma, a}} : \operatorname{Re} a > 0\}$ consists of all composition operators induced by parabolic non-automorphism self-maps of \mathbb{D} fixing γ . If φ is a parabolic non-automorphism self-map of \mathbb{D} , then C_φ is irreducible [11], and $C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$ is a non-zero compact operator (C_φ is essentially normal) [1]. Therefore the unital C^* -algebra $C^*(\mathbb{P}_\gamma)$ is irreducible and $C^*(\mathbb{P}_\gamma) \cap \mathfrak{K} \neq \{0\}$. By Theorem 2.4.9 in [12], $C^*(\mathbb{P}_\gamma)$ contains all compact operators on H^2 . Since the elements of \mathbb{P}_γ satisfy $\rho_{\gamma, a} \circ \rho_{\gamma, b} = \rho_{\gamma, a+b}$, $C^*(\mathbb{P}_\gamma)/\mathfrak{K}$ is a unital commutative C^* -algebra. The following theorem completely describes this C^* -algebra.

Theorem 2.1. [10, Theorem 3 and Corollary 2] *There is a unique $*$ -isomorphism $\Sigma : C([0, 1]) \rightarrow C^*(\mathbb{P}_\gamma)/\mathfrak{K}$ such that $\Sigma(x^a) = [C_{\rho_{\gamma, a}}]$ for $\operatorname{Re} a > 0$. Moreover if ρ is a parabolic non-automorphism self-map of \mathbb{D} fixing γ , then $C^*(C_\rho) = C^*(\mathbb{P}_\gamma)$.*

For a linear-fractional self-map φ of \mathbb{D} ,

$$U_\varphi = C_\varphi (C_\varphi^* C_\varphi)^{-1/2}$$

is the partial isometry in the polar decomposition of C_φ . Since both C_φ and C_φ^* are injective, U_φ is a unitary operator. The following results are useful in finding the C^* -algebra generated by Toeplitz operators and composition operators induced by linear-fractional self-maps of \mathbb{D} .

Theorem 2.2. [9, section 4] *Suppose that φ is a linear-fractional non-automorphism self-map of \mathbb{D} sending $\gamma \in \mathbb{T}$ to $\eta \in \mathbb{T}$. For every $f \in C(\mathbb{T})$ there exists compact operators k and k' such that*

$$T_f C_\varphi = f(\gamma) C_\varphi + k, \quad C_\varphi T_f = f(\eta) C_\varphi + k'.$$

Theorem 2.3. [7] *Let φ_1, φ_2 are automorphisms of \mathbb{D} and $f \in C(\mathbb{T})$. If $U_{\varphi_1}, U_{\varphi_2}$ are unitary parts of the polar decomposition of $C_{\varphi_1}, C_{\varphi_2}$, respectively, then the operators $U_{\varphi_1} U_{\varphi_2} - U_{\varphi_2 \circ \varphi_1}$ and $U_{\varphi_1} T_f U_{\varphi_1}^* - T_{f \circ \varphi_1}$ are compact.*

For $\gamma \in \mathbb{T}$ and non-negative real number t , following [14], we consider the automorphism $\Psi_{\gamma,t}$ of \mathbb{D} defined by

$$\Psi_{\gamma,t}(z) = \frac{(t+1)z + (1-t)}{(1-t)\bar{\gamma}z + (1+t)},$$

which fixes γ , and satisfies $\Psi'_{\gamma,t}(z) = t$. The set $\{\Psi_{\gamma,t} : t > 0\}$ is an abelian group as $\Psi_{\gamma,t_1} \circ \Psi_{\gamma,t_2} = \Psi_{\gamma,t_1 t_2}$ for all $t_1, t_2 > 0$.

If $\varphi, \varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, by equation (4.6) in [14]

$$(2.1) \quad C^*(C_{\varphi_1}, \dots, C_{\varphi_n}, \mathfrak{K}) = C^*(\{C_{\rho_{\gamma,a}} U_{\Psi_{\gamma, \varphi'_1(\gamma)^{m_1} \dots \varphi'_n(\gamma)^{m_n}}} : \operatorname{Re} a > 0, (m_1, \dots, m_n) \in \mathbb{Z}^n\}).$$

In particular

$$(2.2) \quad C^*(C_\varphi, \mathfrak{K}) = C^*(\{C_{\rho_{\gamma,a}} U_{\Psi_{\gamma, \varphi'(\gamma)^n}} : \operatorname{Re} a > 0, n \in \mathbb{Z}\}).$$

Theorem 2.4. [14, Theorem 4.4] *Let G be a collection of automorphisms of \mathbb{D} that fix $\gamma \in \mathbb{T}$. If G is an abelian group and $\eta'(\gamma) \neq 1$ for all $\eta \in G \setminus \{id\}$ then $C^*(\{[C_{\rho_{\gamma,a}} U_\eta] : \operatorname{Re} a > 0, \eta \in G\})$ is $*$ -isomorphic to the minimal unitization of $C_0([0, 1]) \rtimes_\alpha G_d$ where the action $\alpha : G_d \rightarrow \operatorname{Aut}(C_0([0, 1]))$ is defined by $\alpha_\eta(f)(x) = f(x \eta'(\gamma))$ for $\eta \in G$, $f \in C_0([0, 1])$, and $x \in [0, 1]$.*

By (2.1), (2.2) and Theorem 2.4, the C^* -algebras $C^*(C_{\varphi_1}, \dots, C_{\varphi_n}, \mathfrak{K})/\mathfrak{K}$ and $C^*(C_\varphi, \mathfrak{K})/\mathfrak{K}$ are determined as follows.

Corollary 2.5. [14, Theorems 4.6 and 4.7] *Let $\varphi, \varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} that fix $\gamma \in \mathbb{T}$, $\varphi'(\gamma) \neq 1$ and $\ln \varphi'_1, \dots, \ln \varphi'_n$ are linearly independent over \mathbb{Z} . Define the actions $\alpha : \mathbb{Z} \rightarrow \operatorname{Aut}(C_0([0, 1]))$ and $\alpha' : \mathbb{Z}^n \rightarrow \operatorname{Aut}(C_0([0, 1]))$ by $\alpha_n(f)(x) = f(x \varphi'(\gamma)^n)$ and $\alpha'_{(m_1, \dots, m_n)}(f)(x) = f(x \varphi'_1(\gamma)^{m_1} \dots \varphi'_n(\gamma)^{m_n})$, respectively, for $f \in C_0([0, 1])$, $n \in \mathbb{Z}$, $(m_1, \dots, m_n) \in \mathbb{Z}^n$ and $x \in [0, 1]$. Then $C^*(C_\varphi, \mathfrak{K})/\mathfrak{K}$ and $C^*(C_{\varphi_1}, \dots, C_{\varphi_n}, \mathfrak{K})/\mathfrak{K}$ are $*$ -isomorphic to the minimal unitizations of $C_0([0, 1]) \rtimes_\alpha \mathbb{Z}$ and $C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n$, respectively.*

3. LINEAR-FRACTIONAL NON-AUTOMORPHISM SELF-MAPS

Quertermous in [14] shows that if φ is a linear-fractional non-automorphism self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$ then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus C_0([0, 1])$ where $C_\gamma(\mathbb{T})$ is the set of all $f \in C(\mathbb{T})$ vanishing at γ . We extend this result to finitely many composition operators induced by linear-fractional non-automorphism self-maps of \mathbb{D} with a common fixed point on the unit circle. Our approach is similar to that of Quertermous in [14], but there are some complications. As in previous section we use the notation $[T]$ for the coset of T in the Calkin algebra. Let t_1, \dots, t_n be nonzero positive real numbers, $\gamma \in \mathbb{T}$ and Σ be the map defined in Theorem 2.1. Consider

$$\mathcal{N}_{\gamma, t_1, \dots, t_n} = \{\Sigma(g)[U_{\Psi_{\gamma, t_1^{m_1} \dots t_n^{m_n}}}] : g \in C_0([0, 1]), (m_1, \dots, m_n) \in \mathbb{Z}^n\}$$

and let $\mathcal{A}_{\gamma, t_1, \dots, t_n}$ be the non-unital C^* -algebra generated by $\mathcal{N}_{\gamma, t_1, \dots, t_n}$. By Theorem 2.1 and the fact that $\Psi_{\gamma, 1}$ is the identity map of \mathbb{D} , $\mathcal{A}_{\gamma, 1, \dots, 1} \cong C_0([0, 1])$.

Proposition 3.1. *If $\varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, then*

$$C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = C^*(\{[T_f] : f \in C(\mathbb{T})\} \cup \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}).$$

Moreover if $\ln \varphi'_1(\gamma), \dots, \ln \varphi'_n(\gamma)$ are linearly independent over \mathbb{Z} then

$$\mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)} \cong C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n.$$

Proof. First we note that if φ is a linear-fractional non-automorphism self-map of \mathbb{D} fixing $\gamma \in \mathbb{D}$, then $\varphi'(\gamma) > 0$ ([15]). By Theorem 2.1, $\Sigma(x^a) = [C_{\rho_{\gamma, a}}]$ for $\text{Re } a > 0$. Since the closed linear span of $\{x^a : \text{Re } a > 0\}$ is dense in $C_0([0, 1])$, $\mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$ is the same as

$$\begin{aligned} C^*(\{\Sigma(g)[U_{\Psi_{\gamma, \varphi'_1(\gamma)^{m_1} \dots \varphi'_n(\gamma)^{m_n}}}] : g \in C_0([0, 1]), (m_1, \dots, m_n) \in \mathbb{Z}^n\}) \\ = C^*(\{[C_{\rho_{\gamma, a}} U_{\Psi_{\gamma, \varphi'_1(\gamma)^{m_1} \dots \varphi'_n(\gamma)^{m_n}}}] : \text{Re } a > 0, (m_1, \dots, m_n) \in \mathbb{Z}^n\}). \end{aligned}$$

Therefore by (2.1)

$$C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = C^*(\{[T_f] : f \in C(\mathbb{T})\} \cup \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}).$$

The last statement is a consequence of Corollary 2.5. \square

We set

$$\mathcal{C}_{\gamma, t_1, \dots, t_n} = \{[T_f] + A : f \in C^*(\psi), A \in \mathcal{A}_{\gamma, t_1, \dots, t_n}\}.$$

By Proposition 3.1, $C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = C^*(\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)})$. We show that $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$ is indeed a C^* -algebra and describe it. We need the following Lemma.

Lemma 3.2. [14, Lemma 6.3] *If $\gamma \in \mathbb{T}$, $f \in C(\mathbb{T})$ and $A \in \mathcal{A}_{\gamma, t_1, \dots, t_n}$, then*

$$[T_f]A = f(\gamma)A = A[T_f].$$

Moreover if $[T_f] + A = [0]$ then $f \equiv 0$ and $A = 0$.

The first part of above lemma says that $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$ is closed under multiplication and it is a dense $*$ -subalgebra of $C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$.

Theorem 3.3. *If $\varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$, then $C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$.*

Proof. Let \mathcal{B} be the minimal unitization of $C_\gamma(\mathbb{T}) \oplus \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$. Since $C_\gamma(\mathbb{T}) = \{f - f(\gamma) : f \in C(\mathbb{T})\}$ we may define the map $\Delta : \mathcal{B} \rightarrow \mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$ by

$$\Delta((f - f(\gamma), A) + f(\gamma)I) = [T_f] + A,$$

for $f \in C(T)$ and $A \in \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$. By part two of Lemma 3.2, Δ is injective. Let $\alpha = (f_1 - f_1(\gamma), A_1) + f_1(\gamma)I$ and $\beta = (f_2 - f_2(\gamma), A_2) + f_2(\gamma)I$, for some $f_1, f_2 \in C(T)$ and $A_1, A_2 \in \mathcal{A}$. Then

$$\begin{aligned} \Delta(\alpha\beta) &= \Delta(f_1f_2 - f_1(\gamma)f_2 - f_2(\gamma)f_1 + f_1(\gamma)f_2(\gamma), A_1A_2) \\ &\quad + f_1(\gamma)(f_2 - f_2(\gamma), A_2) + f_2(\gamma)(f_1 - f_1(\gamma), A_1) + f_1(\gamma)f_2(\gamma) \\ &= \Delta(f_1f_2 - f_1(\gamma)f_2(\gamma), A_1A_2 + f_1(\gamma)A_1 + f_2(\gamma)A_2) + f_1(\gamma)f_2(\gamma)I \\ &= [T_{f_1f_2}] + A_1A_2 + f_1(\gamma)A_2 + f_2(\gamma)A_1. \end{aligned}$$

Therefore, again by Lemma 3.2,

$$\Delta(\alpha\beta) = [T_{f_1f_2}] + A_1A_2 + [T_{f_1}]A_2 + [T_{f_2}]A_1 = ([T_{f_1}] + A_1)([T_{f_2}] + A_2) = \Delta(\alpha)\Delta(\beta).$$

Hence Δ is an injective $*$ -homomorphism, and thus an isometry with closed range $\mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$. Therefore the C^* -algebra

$$C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K} = \mathcal{C}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$$

is $*$ -isomorphic to the minimal unitization of $C_\gamma(\mathbb{T}) \oplus \mathcal{A}_{\gamma, \varphi'_1(\gamma), \dots, \varphi'_n(\gamma)}$. \square

The following result immediately follows from the above theorem and Proposition 3.1.

Corollary 3.4. *If $\varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$ and $\ln \varphi'_1(\gamma), \dots, \ln \varphi'_n(\gamma)$ are linearly independent over \mathbb{Z} , then $C^*(T_z, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of the direct sum $C_\gamma(\mathbb{T}) \oplus (C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n)$.*

4. IRREDUCIBLE TOEPLITZ OPERATORS

Let X be a compact Hausdorff space and \mathcal{A} be a C^* -subalgebra of $C(X)$ containing the constants. For $x, y \in X$, put $x \sim y$ if and only if $f(x) = f(y)$ for all f in \mathcal{A} . Then \sim is an equivalence relation on X . Let $[x]$ denote the equivalence class of x and $[X]$ be the quotient space and equip $[X]$ with the weak topology induced by \mathcal{A} . Let X/\sim be the quotient space equipped with the quotient topology. Then \mathcal{A} is $*$ -isomorphic to $C([X])$ and a C^* -subalgebra of $C(X/\sim)$ via $f \mapsto \tilde{f}$ where $\tilde{f}([x]) := f(x)$ for $x \in X$. Note that $[X]$ is always Hausdorff and it is homeomorphic to X/\sim when the latter is Hausdorff [5].

If D is an irreducible C^* -subalgebra of $C^*(T_z)$, then it contains a nonzero compact operator [5]. Hence D contains all of compact operators on H^2 [12]. We set

$$D_0 = \{f \in C(\mathbb{T}) : T_f \in D\}.$$

Theorem 4.1. [5, Theorem 2 and 5] *If D is an irreducible C^* -subalgebra of $C^*(T_z)$, then D_0 is a C^* -subalgebra of $C(\mathbb{T})$ and D/\mathfrak{K} is $*$ -isomorphic to $C([\mathbb{T}])$, where $[\mathbb{T}]$ is the quotient with respect to the equivalence relation induced by D_0 .*

Let T_ψ be an irreducible Toeplitz operator (i.e. the only closed vector subspaces of H^2 reducing for T_ψ are 0 and H^2) with continuous symbol ψ (see [2, 4, 5, 13]). Then $D = C^*(T_\psi)$ is irreducible and by Theorem 6 in [5], D_0 is the C^* -algebra of $C(\mathbb{T})$ generated by ψ . Hence by Theorem 4.1, D/\mathfrak{K} is $*$ -isomorphic to $C([\mathbb{T}])$ where $[\mathbb{T}]$ is the quotient with respect to the equivalence relation induced by ψ (that is $x \sim y$ if and only if $\psi(x) = \psi(y)$).

Note that T_z is irreducible and there are other irreducible Toeplitz operators (for example, see Example 1 and 2 in [13]). If $D = C^*(T_\psi) = C^*(T_z)$ For some continuous function ψ , then $D_0 = C(\mathbb{T})$ is generated by ψ and by the Stone-Weierstrass theorem, ψ must be one-to-one on the unit circle. Therefore we are interested in the case that ψ is not one-to-one on \mathbb{T} .

Let T_ψ be an irreducible Toeplitz operator and $[\mathbb{T}]$ be the quotient space with respect to the equivalence relation induced by ψ . Put

$$C_{[\gamma]}([\mathbb{T}]) := \{f \in C([\mathbb{T}]) : f([\gamma]) = 0\}$$

and let $\mathcal{B}_{\gamma,t}$ be the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus \mathcal{A}_{\gamma,t}$.

Theorem 4.2. *If T_ψ is an irreducible Toeplitz operator on Hardy space H^2 with symbol $\psi \in C(\mathbb{T})$ and φ is a linear-fractional non-automorphism self-map of \mathbb{T} fixing $\gamma \in \mathbb{T}$, then $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to $\mathcal{B}_{\gamma, \varphi'(\gamma)}$.*

Proof. Similar to the proof of Proposition 3.1,

$$(4.1) \quad C^*(T_\psi, C_\varphi)/\mathfrak{K} = C^*(\{[T_\phi] : \phi \in C^*(\psi)\} \cup \mathcal{A}_{\gamma, \varphi'(\gamma)}).$$

For $t > 0$, set

$$\mathcal{C}_{\gamma,t,\psi} := \{[T_\phi] + A : \phi \in C^*(\psi), A \in \mathcal{A}_{\gamma,t}\}.$$

By (4.1), $C^*(T_\psi, C_\varphi)/\mathfrak{K} = C^*(\mathcal{C}_{\gamma,t,\psi})$. We show that $\mathcal{C}_{\gamma,t,\psi}$ is a C^* -algebra and describe it. It is clear that $\mathcal{C}_{\gamma,t,\psi}$ is closed under taking linear combination and adjoint. On the other hand, by Lemma 3.2 and the fact that for $\phi_1, \phi_2 \in C^*(\psi)$, $[T_{\phi_1}][T_{\phi_2}] = [T_{\phi_1\phi_2}]$ (since $T_{\phi_1\phi_2} - T_{\phi_1}T_{\phi_2}$ is a compact operator), $\mathcal{C}_{\gamma,t,\psi}$ is also closed under multiplication. Hence $\mathcal{C}_{\gamma,t,\psi}$ is a dense $*$ -subalgebra of $C^*(T_\psi, C_\varphi)/\mathfrak{K}$. Similar to the proof of Theorem 3.3, we define the map $\mathcal{F} : \mathcal{B}_{\gamma, \varphi'(\gamma)} \rightarrow \mathcal{C}_{\gamma, \varphi'(\gamma), \psi}$, by

$$\mathcal{F}((\tilde{f} - \tilde{f}([\gamma]), A) + \tilde{f}([\gamma])I = [T_f] + A,$$

for $f \in C^*(\psi)$ and $A \in \mathcal{A}_{\gamma, \varphi'(\gamma)}$. By Lemma 3.2, \mathcal{F} is an injective $*$ -homomorphism and its image is $\mathcal{C}_{\gamma, \varphi'(\gamma), \psi} = C^*(T_\psi, C_\varphi)/\mathfrak{K}$. \square

The following results are straightforward consequences of Theorems 2.1, 4.2 and Corollary 2.5.

Corollary 4.3. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and ρ is a parabolic non-automorphism self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$ then, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus C_0([0, 1])$.*

Corollary 4.4. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and φ is linear-fractional non-automorphism self-map of \mathbb{D} fixing $\gamma \in \mathbb{T}$ such that $\varphi'(\gamma) \neq 1$, then $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus (C_0([0, 1]) \rtimes_\alpha \mathbb{Z})$, where the action α is defined as in Corollary 2.5.*

Corollary 4.5. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and $\varphi_1, \dots, \varphi_n$ are linear-fractional non-automorphism self-maps of \mathbb{D} fixing $\gamma \in \mathbb{T}$ such that $\ln \varphi_1'(\gamma), \dots, \ln \varphi_n'(\gamma)$ are linearly independent over \mathbb{Z} , then $C^*(T_\psi, C_{\varphi_1}, \dots, C_{\varphi_n})/\mathfrak{K}$ is $*$ -isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus (C_0([0, 1]) \rtimes_{\alpha'} \mathbb{Z}^n)$.*

Now consider the case that φ is a linear-fractional non-automorphism self-map of \mathbb{D} such that $\varphi(\gamma) = \eta$ for some $\varphi \neq \eta \in \mathbb{T}$. Kriete, MacCluer and Moorhouse investigated this case in [9]. We summarize their results as follows.

Theorem 4.6. [9] *Let φ be a linear-fractional non-automorphism self-map of \mathbb{D} with $\varphi(\gamma) = \eta$ for some $\varphi \neq \eta \in \mathbb{T}$. Then for every $a \in \mathcal{A} = C^*(T_z, C_\varphi)/\mathfrak{K}$ there is a unique $\omega \in C(\mathbb{T})$ and unique functions f, g, h and k in $C_0([0, 1])$ such that*

$$a = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*]).$$

Moreover the map $\Phi : \mathcal{A} \rightarrow C(\mathbb{T}) \oplus \mathbb{M}_2(C([0, 1]))$ defined by

$$\Phi(a) = \left(\omega, \begin{bmatrix} \omega(\gamma) + g & h \\ k & \omega(\eta) + f \end{bmatrix} \right)$$

is a $*$ -isomorphism of \mathcal{A} onto the following C^* -subalgebra of $C(\mathbb{T}) \oplus \mathbb{M}_2(C([0, 1]))$

$$\mathcal{D} = \left\{ (\omega, V) \in C(\mathbb{T}) \oplus \mathbb{M}_2(C([0, 1])) : V(0) = \begin{bmatrix} \omega(\gamma) & 0 \\ 0 & \omega(\eta) \end{bmatrix} \right\}.$$

We replace the shift operator by an arbitrary irreducible Toeplitz operator T_ψ with continuous symbol.

Theorem 4.7. *Let φ be a linear-fractional non-automorphism self-map of \mathbb{D} such that $\varphi(\gamma) = \eta$ for distinct points $\gamma, \eta \in \mathbb{T}$ and T_ψ be irreducible with continuous symbol ψ on \mathbb{T} . Then every element b in $\mathcal{B} = C^*(T_\psi, C_\varphi)/\mathfrak{K}$ has a unique representation of the form*

$$b = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*])$$

where $\omega \in C^*(\psi)$ and f, g, h and k are in $C_0([0, 1])$. Moreover \mathcal{B} is $*$ -isomorphic to the C^* -subalgebra \mathcal{D} of $C([\mathbb{T}]) \oplus M_2(C([0, 1]))$ defined by

$$\mathcal{D} = \left\{ (f, S) \in C([\mathbb{T}]) \oplus \mathbb{M}_2(C([0, 1])) : S(0) = \begin{bmatrix} f([\gamma]) & 0 \\ 0 & f([\eta]) \end{bmatrix} \right\}.$$

Proof. Since \mathcal{B} is a C^* -subalgebra of $C^*(T_z, C_\varphi)/\mathfrak{K}$, by Theorem 4.6, for every element $b \in \mathcal{B}$ there is a unique $\omega \in C(\mathbb{T})$ and unique functions f, g, h and k in $C_0([0, 1])$ such that

$$b = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*]).$$

We show that $\omega \in C^*(\psi)$. By Theorem 2.2, for each $\varepsilon > 0$, there is an element $b_\varepsilon \in \mathcal{B}$ such that $\|b_\varepsilon - b\| < \varepsilon$ and $b_\varepsilon = p([T_\psi], [T_\psi^*]) + q([C_\varphi], [C_\varphi^*])$, for some polynomials p and q . It is straightforward to show that $p([T_\psi], [T_\psi^*]) = [T_{p(\psi, \bar{\psi})}]$. Hence by Theorem 4.6, there are unique functions $f_\varepsilon, g_\varepsilon, h_\varepsilon$ and k_ε in $C_0([0, 1])$ such that

$$b_\varepsilon - b = [T_{\omega - p(\psi, \bar{\psi})}] + f_\varepsilon([C_\varphi^* C_\varphi]) + g_\varepsilon([C_\varphi C_\varphi^*]) + [U_\varphi]h_\varepsilon([C_\varphi^* C_\varphi]) + [U_\varphi^*]k_\varepsilon([C_\varphi C_\varphi^*])$$

and

$$\left\| \left(\omega - p(\psi, \bar{\psi}), \begin{bmatrix} \omega(\gamma) + g_\varepsilon & h_\varepsilon \\ k_\varepsilon & \omega(\eta) + f_\varepsilon \end{bmatrix} \right) \right\| = \|b_\varepsilon - b\| < \varepsilon.$$

Thus $\|\omega - p(\psi, \bar{\psi})\| < \varepsilon$ and $\omega \in C^*(\psi)$. By Theorem 4.6, \mathcal{B} is $*$ -isomorphic to the following C^* -subalgebra of $C^*(\psi) \oplus M_2(C([0, 1]))$

$$\mathcal{C} = \left\{ (\omega, S) \in C^*(\psi) \oplus M_2(C([0, 1])) : S(0) = \begin{bmatrix} \omega(\gamma) & 0 \\ 0 & \omega(\eta) \end{bmatrix} \right\}.$$

Hence by Theorem 4.1, \mathcal{B} is $*$ -isomorphic to

$$\mathcal{D} = \left\{ (f, S) \in C([T]) \oplus M_2(C([0, 1])) : S(0) = \begin{bmatrix} f([\gamma]) & 0 \\ 0 & f([\eta]) \end{bmatrix} \right\}.$$

□

5. COMPOSITION OPERATORS WITH AUTOMORPHISM SYMBOLS

A self-map φ of the unit disk \mathbb{D} is an automorphism if φ is a one-to-one holomorphic map of \mathbb{D} onto \mathbb{D} . We denote the class of automorphisms of \mathbb{D} by $Aut(\mathbb{D})$. A well-know consequence of Schwarz Lemma shows that every element $\varphi \in Aut(\mathbb{D})$ has the form

$$(5.1) \quad \varphi(z) = \omega \frac{s - z}{1 - \bar{s}z},$$

for some $\omega \in \mathbb{T}$, where $s = \varphi^{-1}(0) \in \mathbb{D}$.

A *Fuchsian group* Γ is a discrete group of automorphisms of \mathbb{D} . Fix a point $z_0 \in \mathbb{D}$. The limit set of Γ is the set of limit points of the orbit $\{\varphi(z_0) : \varphi \in \Gamma\}$ in \mathbb{D} . This is a closed subset of the unit circle and does not depend on the choice of z_0 . The limit set of a Fuchsian group has either 0, 1, 2, or infinitely many elements. When the limit set is infinite, it is perfect and nowhere dense (hence uncountable). A Fuchsian group Γ is called *non-elementary* if its limit set is infinite.

Jury in [7] describes the C^* -algebra $C^*(\{C_\varphi : \varphi \in \Gamma\})/\mathfrak{K}$ when Γ is a non-elementary Fuchsian group. A basic point in the proof is that the non-elementary condition on Γ guarantees that the C^* -algebra $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift T_z . In the next proposition we weaken this condition on Γ and still get the shift operator. We need the following lemma.

Lemma 5.1. [7, Lemma 3.2] *If φ is an automorphism of \mathbb{D} with $a = \varphi^{-1}(0)$ and*

$$f(z) = \frac{1 - \bar{a}z}{(1 - |a|^2)^{1/2}},$$

then $C_\varphi C_\varphi^ = T_z T_z^*$.*

Proposition 5.2. *Let Γ be a group of automorphisms on \mathbb{D} . If the orbit $\{\varphi(0) : \varphi \in \Gamma\}$ consists of at least two linear independent points over \mathbb{R} , then $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift T_z .*

Proof. Let φ_1, φ_2 are two distinct elements of Γ . If $a_1 = \varphi_1(0)$ and $a_2 = \varphi_2(0)$, then by Lemma 5.1,

$$(1 - |a_i|^2)C_{\varphi_i^{-1}}C_{\varphi_i}^* = (I - \bar{a}_i T_z)(I - \bar{a}_i T_z)^*,$$

for $i = 1, 2$. Therefore the unital C^* -algebra $\mathcal{D} := C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains

$$s_i = \bar{a}_i T_z + a_i T_z^* - |\bar{a}_i|^2 T_z T_z^*, \quad (i = 1, 2).$$

If $a_1 = \alpha + i\beta, a_2 = \alpha' + i\beta'$ then a simple computation shows that

$$\det \begin{pmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{pmatrix} = -2i \det \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}.$$

Hence if a_1, a_2 are linearly independent over \mathbb{R} then the linear system

$$\begin{pmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has a unique solution. Therefore a linear combination of s_1, s_2 is of the form $T_z^* + tT_zT_z^*$, for some scalar $t \in \mathbb{C}$. If $t = 0$, then \mathcal{D} contains T_z , otherwise, since

$$I + tT_z = (T_z^* + tT_zT_z^*)(T_z^* + tT_zT_z^*)^* - \bar{t}(T_z^* + tT_zT_z^*) \in \mathcal{D},$$

again $T_z \in \mathcal{D}$. \square

Corollary 5.3. *Let Γ be a Fuchsian group. If the limit set of Γ contains at least two linearly independent points, then $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains the unilateral shift operator.*

Proof. If z_1, z_2 are linearly independent points of the limit set, then one can choose two elements of the orbit $\{\varphi(0) : \varphi \in \Gamma\}$ near to z_1, z_2 that are linearly independent. Hence by Proposition 5.2, the above C^* -algebra contains the unilateral shift operator. \square

As a consequence if Γ be a non-elementary Fuchsian group, then the above corollary shows that $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains T_z . Since the action of a Fuchsian group is amenable and topologically free, we have the following result, which could be proved by Proposition 5.2, similar to Theorem 3.1 in [7].

Corollary 5.4. *Let Γ be a Fuchsian group and the orbit $\{\varphi(0) : \varphi \in \Gamma\}$ consists of at least two linear independent points over \mathbb{R} . Then there is an exact sequence*

$$0 \rightarrow \mathfrak{K} \rightarrow C^*(\{C_\varphi : \varphi \in \Gamma\}) \rightarrow C(\mathbb{T}) \rtimes \Gamma \rightarrow 0.$$

As an example of this situation, let Γ be a Fuchsian group and contains a hyperbolic map (a linear-fractional map with two distinct fixed points on the unit circle) whose fixed points are not diagonal (endpoints of one diameter). Then the C^* -algebra $C^*(\{C_\varphi : \varphi \in \Gamma\})$ contains T_z . This follows from the fact that fixed points of hyperbolic and parabolic maps are contained in the limit set.

Corollary 5.5. *Let φ is an automorphism of the unit disk. If $\varphi^n(0)$ and $\varphi^m(0)$ are linearly independent over \mathbb{R} , for some $n, m \in \mathbb{Z}$, then the unital C^* -algebra $C^*(C_\varphi)$ contains the shift operator.*

Proof. Since $C_{\varphi^{-1}}$ is the inverse of C_φ , the C^* -algebra $C^*(C_\varphi)$ contains C_{φ^n} , for all $n \in \mathbb{Z}$. Now the result follows from Proposition 5.2. \square

Let $\varphi \in \text{Aut}(\mathbb{D})$ be of the form $\varphi(z) = \omega \frac{s-z}{1-\bar{s}z}$ for some $\omega \in \mathbb{T}$ and $s \in \mathbb{D}$. If ω is not real ($\omega \neq \pm 1$) and $s \neq 0$ then $T_z \in C^*(C_\varphi)$. Indeed, a simple calculation shows that

$$\varphi(0) = \omega s, \quad \varphi^2(0) = \omega s \frac{1-\omega}{1-|s|^2\omega}$$

and if $\omega = x + iy$ ($y \neq 0$), then

$$\frac{1-\omega}{1-|s|^2\omega} = \frac{1+|s|^2(1-x)+i(|s|^2y-y)}{|1-|s|^2\omega|^2}.$$

Therefore $\frac{1-\omega}{1-|s|^2\omega}$ is not real and $\varphi(0)$ and $\varphi^2(0)$ are linearly independent over \mathbb{R} . Hence Corollary 5.5 shows that $T_z \in C^*(C_\varphi)$. If ω is real (1 or -1) or $s \neq 0$, then all $\varphi^n(0)$'s are dependent.

Jury in [8] finds the C^* -algebra $C^*(T_z, C_\varphi)/\mathfrak{K}$, for $\varphi \in \text{Aut}(\mathbb{D})$, as a crossed product C^* -algebra. We do the same when the shift operator is replaced by a general irreducible Toeplitz operator T_ψ . The above example shows that if $\varphi \in \text{Aut}(\mathbb{D})$ be of the form (5.1) for some non-real $\omega \in \mathbb{T}$ and non-zero $s \in \mathbb{D}$, then the structure of $C^*(T_z, C_\varphi)/\mathfrak{K} = C^*(C_\varphi)/\mathfrak{K}$ does not change, if one replaces T_z with T_ψ . Here we check the case $s = 0$.

Theorem 5.6. *Let T_ψ is irreducible and $\varphi \in \text{Aut}(\mathbb{D})$. If the composition operator C_φ is unitary and $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$, then there is an exact sequence of C^* -algebras*

$$0 \rightarrow \mathfrak{K} \rightarrow C^*(T_\psi, C_\varphi) \rightarrow C(\psi(\mathbb{T})) \rtimes_\varphi \mathbb{Z} \rightarrow 0.$$

Proof. The image $X = \psi(\mathbb{T})$ of ψ is a compact Hausdorff space and $\varphi^n(X) = X$, for all $n \in \mathbb{Z}$. Now \mathbb{Z} acts on X by

$$\begin{aligned} \beta : \mathbb{Z} &\rightarrow \text{Home}(X) \\ n &\mapsto \beta_n, \quad \beta_n(x) = \varphi^n(x), \end{aligned}$$

for $n \in \mathbb{Z}$ and $x \in X$. This induces an action of \mathbb{Z} on $C(X)$ given by

$$\begin{aligned} \alpha : \mathbb{Z} &\rightarrow \text{Aut}(C(X)) \\ \alpha_n(f)(x) &= f(\varphi^{-n}(x)). \end{aligned}$$

The C^* -algebra $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is generated by $C^*(T_\psi)/\mathfrak{K} \cong C(X)$ and unitaries $[C_{\varphi^n}]$. On the other hand, Theorem 2.3 shows that the unitary representation $n \rightarrow [C_{\varphi^{-n}}]$ satisfies the covariance relation $[C_\varphi]f[C_\varphi^*] = \alpha_n(f)$. Hence there is a surjective $*$ -homomorphism from the full crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ to $C^*(T_\psi, C_\varphi)/\mathfrak{K}$. But the action of the amenable group \mathbb{Z} on compact Hausdorff space X is amenable and topologically free (i.e. for each $n \in \mathbb{Z}$, the set of points that are fixed by φ^n has empty interior) thus similar to the proof of Theorem 2.1 in [8], the above $*$ -homomorphism is also injective and hence an isometry. \square

Corollary 5.7. *Let $\varphi \in \text{Aut}(\mathbb{D})$ be of the form (5.1) with $s = 0$. If T_ψ is irreducible and $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$, then there is an exact sequence of C^* -algebras*

$$0 \rightarrow \mathfrak{K} \rightarrow C^*(T_\psi, C_\varphi) \rightarrow C(\psi(\mathbb{T})) \rtimes_\varphi \mathbb{Z} \rightarrow 0.$$

Proof. In this case $\varphi(z) = \omega z$, for some $\omega \in \mathbb{T}$. It is easy to check that C_φ is an isometry from H^2 onto H^2 . Hence by the above theorem, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is isomorphic to the crossed product $C(\psi(\mathbb{T})) \rtimes_\varphi \mathbb{Z}$. \square

One may wonder if for $\varphi \in \text{Aut}(\mathbb{D})$ of the form (5.1) with $s = 0$, there is a function ψ that satisfies in the hypothesis of the above corollary. For example, let $\varphi(z) = ze^{i\frac{2\pi}{3}}$. Theorem 1 in [13] gives a sufficient condition for irreducibility of Toeplitz operator T_ψ : if the restriction of a function $\psi \in H^2$ to a Borel subset $S \subseteq \mathbb{T}$, with positive normalized Haar measure on the unit circle, is one-to-one and the sets $\psi(S)$ and $\psi(\mathbb{T} \setminus S)$ are disjoint, then T_ψ is irreducible. By the Riemann mapping theorem (for example, see [6]) there exists a biholomorphic (bijective and holomorphic) map ψ from the unit disk \mathbb{D} onto the simply connected set

$$\mathbb{D} - ([1/2, 1) \cup [-1/4 + \sqrt{3}/4i, -1/2 + \sqrt{3}/2i) \cup [-1/4 - \sqrt{3}/4i, -1/2 - \sqrt{3}/2i))$$

in the complex plane. A result of Carathéodory in [3] states that ψ extends continuously to the closure of the unit disk. Moreover one can choose ψ (see Figure 1) such that

$$\psi(A) = A', \psi(B) = B', \psi(C) = C', \psi(K) = K, \psi(K') = K', \psi(K'') = K'',$$

and

$$\psi(D) = \psi(D') = A, \psi(E) = \psi(E') = B, \psi(F) = \psi(F') = C.$$

where capital letters show points in the closure of the unit disk. The restriction of ψ to \mathbb{T} , also denoted by ψ , is an element of H^2 , one-to-one on

$$S = \{e^{i\theta} : \theta \in (\frac{2\pi}{15}, \frac{8\pi}{15}) \cup (\frac{4\pi}{5}, \frac{6\pi}{5}) \cup (\frac{22\pi}{15}, \frac{28\pi}{15})\},$$

$\psi(S)$ and $\psi(\mathbb{T} \setminus S)$ are disjoint and the image of ψ is invariant under φ , that is, $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$. Moreover since ψ is not one-to-one on \mathbb{T} , $C^*(T_\psi) \neq C^*(T_z)$.

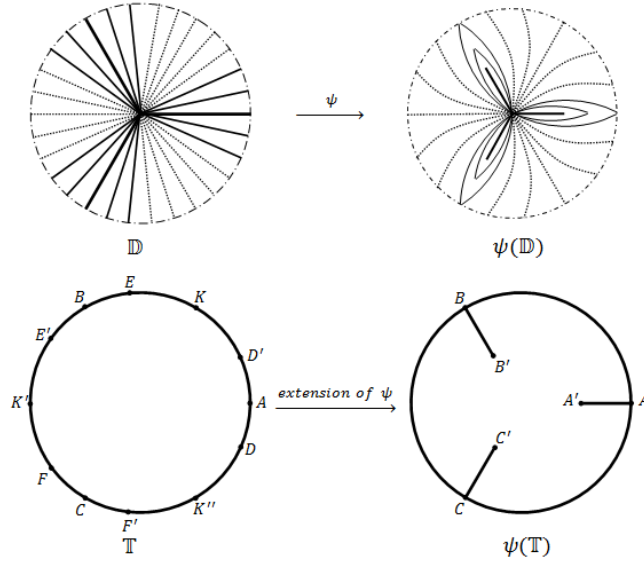


FIGURE 1.

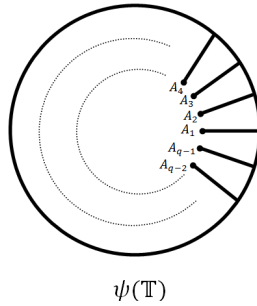


FIGURE 2.

More generally, if the automorphism φ is of the form $\varphi(z) = ze^{i\frac{2p}{q}\pi}$ where p and q are relatively prime integers with q positive, then by a similar construction,

there is a function ψ that satisfies the conditions of the above corollary and is not one-to-one on the unit circle and

$$\psi(\mathbb{T}) = \mathbb{T} \cup \left(\bigcup_{n=0}^{n=q-1} \varphi^n([1/2, 1)) \right),$$

(see Figure 2). We show that the C^* -algebras $C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}$ and $C(\mathbb{T}) \rtimes_{\varphi} \mathbb{Z}$ are non isomorphic. This is of course easy when $p = 0$, that is, $\varphi(z) = z$ on \mathbb{T} and the action of \mathbb{Z} on \mathbb{T} and $\psi(\mathbb{T})$ is trivial. In this case,

$$C(\psi(\mathbb{T})) \rtimes_{id} \mathbb{Z} \cong C(\psi(\mathbb{T})) \otimes C^*(\mathbb{Z}) \cong C(\psi(\mathbb{T})) \otimes C(\mathbb{T}) \cong C(\psi(\mathbb{T}) \times \mathbb{T})$$

and $C(\mathbb{T}) \rtimes_{id} \mathbb{Z} \cong C(\mathbb{T} \times \mathbb{T})$. The spectrum of these C^* -algebras are $\psi(\mathbb{T}) \times \mathbb{T} = (\mathbb{T} \cup [1/2, 1)) \times \mathbb{T}$ and \mathbb{T}^2 , which are not homeomorphic: if, on the contrary, Φ is a homeomorphism from $(\mathbb{T} \cup [1/2, 1)) \times \mathbb{T}$ onto \mathbb{T}^2 and S is the half of \mathbb{T} on the left side of the imaginary axis, then $\Phi(S \times \mathbb{T}) = X \times Y$ is connected, and so are X and Y . If $X \neq \mathbb{T}$ and $Y \neq \mathbb{T}$ then $\mathbb{T} \setminus X$ and $\mathbb{T} \setminus Y$ are connected subsets of \mathbb{T} . Hence $((\mathbb{T} \setminus X) \times \mathbb{T}) \cup (\mathbb{T} \times (\mathbb{T} \setminus Y)) \subseteq \mathbb{T}^2$ is connected, and so is $\Phi^{-1}(((\mathbb{T} \setminus X) \times \mathbb{T}) \cup (\mathbb{T} \times (\mathbb{T} \setminus Y))) = (\psi(\mathbb{T}) \setminus S) \times \mathbb{T}$, a contradiction, as $\psi(\mathbb{T}) \setminus S$ is not connected. Similarly $X = \mathbb{T}$ or $Y = \mathbb{T}$ lead to a contradiction.

For the more general example, we need some preparation to show that the crossed products are non isomorphic (we refer the reader to [16] for more details). Let Y is a topological space. The T_0 -ization of Y is the quotient space $(Y)^{\sim} = Y / \sim$ where \sim is the equivalence relation on Y defined by $x \sim y$ if $\overline{\{x\}} = \overline{\{y\}}$. The space $(Y)^{\sim}$, equipped with the quotient topology, is a T_0 topological space (see Lemma 6.10 in [16]).

Let G be a topological group acting on a topological space X from left. The orbit of $x \in X$ is the set $G \cdot x = \{s \cdot x : s \in G\}$. The stability group at x is $G_x := \{s \in G : s \cdot x = x\}$. The set of orbits is denoted by $G \backslash X$ and is called the orbit space. It is equipped with the weakest topology making the natural quotient map $p : X \rightarrow G \backslash X$ continuous.

By Lemma 3.35 in [16], if X is second countable or locally compact, then so is $G \backslash X$. When G is a locally compact abelian group and \widehat{G} is the character group of G , one could define an equivalence relation on $X \times \widehat{G}$ by $(x, \tau) \sim (y, \sigma)$ if $\overline{G \cdot x} = \overline{G \cdot y}$ and $\tau \bar{\sigma} \in G_x^{\perp}$. The space $X \times \widehat{G} / \sim$ is given the quotient topology. Note that if X is a T_0 -space, then $X \times \widehat{G} / \sim = X \times \widehat{G} / G_x^{\perp}$. By Remark 8.42 in [16], $X \times \widehat{G} / \sim$ is homeomorphic to $(G \backslash X)^{\sim} \times \widehat{G} / \sim$.

Proposition 5.8. *Let ψ be as above and $\varphi(z) = ze^{i\frac{2p}{q}\pi}$, with p and q relatively prime integers and $q > 0$. Then the spectrum of the C^* -algebra $C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}$ is homeomorphic to $(\mathbb{T} \cup [1/2, 1)) \times \mathbb{T}$.*

Proof. Consider the map

$$f : \mathbb{Z} \backslash \psi(\mathbb{T}) \rightarrow \mathbb{T} \cup [1/2, 1)$$

$$\mathbb{Z} \cdot z \mapsto \begin{cases} z^q & \text{if } z \in \mathbb{T}, \\ z^q / |z|^{q-1} & \text{otherwise.} \end{cases}$$

Since the map

$$g : \psi(\mathbb{T}) \rightarrow \mathbb{T} \cup [1/2, 1)$$

$$z \mapsto \begin{cases} z^q & \text{if } z \in \mathbb{T}, \\ z^q / |z|^{q-1} & \text{otherwise} \end{cases}$$

is continuous, $g^{-1}(U)$ is open in $\psi(\mathbb{T})$ for each open set U in $\mathbb{T} \cup [1/2, 1)$. Since the quotient map $p : \psi(\mathbb{T}) \rightarrow \mathbb{Z} \backslash \psi(\mathbb{T})$ is open (Lemma 3.25 in [16]), $f^{-1}(U) = p(g^{-1}(U))$ is open in $\mathbb{Z} \backslash \psi(\mathbb{T})$ and so f is continuous. On the other hand, $\mathbb{Z} \backslash \psi(\mathbb{T})$ is compact, therefore f is a homeomorphism. Also $\mathbb{Z} \backslash \psi(\mathbb{T})$ is T_0 and \mathbb{Z} and $\psi(\mathbb{T})$ are second countable, hence by Theorem 8.39 and Remark 8.42 in [16],

$$\begin{aligned} \text{Prim}(C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}) &\cong (\mathbb{Z} \backslash \psi(\mathbb{T}))^{\sim} \times \widehat{\mathbb{Z}} / \sim \cong \mathbb{Z} \backslash \psi(\mathbb{T}) \times \widehat{\mathbb{Z}} / (q\mathbb{Z})^{\perp} \\ &\cong (\mathbb{T} \cup [1/2, 1)) \times \widehat{q\mathbb{Z}} \cong (\mathbb{T} \cup [1/2, 1)) \times \mathbb{T}. \end{aligned}$$

But all \mathbb{Z} -orbits $\mathbb{Z} \cdot z$ are finite (and so closed in $\psi(\mathbb{T})$), therefore by Theorem 8.44 in [16], $C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}$ is a liminal C^* -algebra and by Theorem 5.6.4 in [12], $\text{Prim}(C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z})$ is homeomorphic to the spectrum $(C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z})^{\wedge}$, as required. \square

Now we could finish the argument for the general example. The spectrums of the C^* -algebras $C(\psi(\mathbb{T})) \rtimes_{\varphi} \mathbb{Z}$ and $C(\mathbb{T}) \rtimes_{\varphi} \mathbb{Z}$ are $\psi(\mathbb{T}) \times \mathbb{T}$ and \mathbb{T}^2 , which are not homeomorphic. Hence these C^* -algebras are not isomorphic.

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